

The University of Chicago

CONGRUENCES
DETERMINED BY A GIVEN SURFACE

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE ODGEN GRADUATE SCHOOL
OF SCIENCE IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY
CLARIBEL KENDALL

Private Edition, Distributed By
The University of Chicago Libraries
Chicago, Illinois

Reprinted from
AMERICAN JOURNAL OF MATHEMATICS
Vol. XLV, No. 1, January, 1923

The University of Chicago

CONGRUENCES
DETERMINED BY A GIVEN SURFACE

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE ODGEN GRADUATE SCHOOL
OF SCIENCE IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS



BY
CLARIBEL KENDALL

Private Edition, Distributed By
The University of Chicago Libraries
Chicago, Illinois

Reprinted from
AMERICAN JOURNAL OF MATHEMATICS
Vol. XLV, No. 1, January, 1923

Digitized by the Internet Archive
in 2026

CONGRUENCES DETERMINED BY A GIVEN SURFACE.

BY CLARIBEL KENDALL.

§ 1. Introduction.

It has been shown by Professor Wilczynski* that a non-developable analytic surface S may be regarded as an integrating surface of a non-involutory, completely integrable system of partial differential equations of the form

$$(1) \quad \begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned}$$

where the subscripts denote partial differentiation, and where the coefficients, which are seminvariants, are analytic functions of u and v satisfying the integrability conditions

$$(2) \quad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0. \end{aligned}$$

Then the curves $u = \text{const.}$, $v = \text{const.}$ form an asymptotic net on the surface S . We shall assume that in general $a' \neq 0$, $b \neq 0$, thus excluding ruled surfaces from our discussion.†

Under the above conditions (1) has exactly four linearly independent solutions

$$(3) \quad y^{(k)} = f^{(k)}(u, v) \quad (k = 1, 2, 3, 4),$$

which are interpreted as the homogeneous coördinates of a point y on the surface S . The semicovariants of (1) are‡

$$(4) \quad y, \quad y_u, \quad y_v, \quad y_{uv}.$$

Substituting the values (3) for y in (4) we obtain four points y, y_u, y_v, y_{uv} which are not coplanar since no relations of the form

$$\alpha y^{(k)} + \beta y_u^{(k)} + \gamma y_v^{(k)} + \delta y_{uv}^{(k)} = 0 \quad (k = 1, 2, 3, 4)$$

can exist among them. For, otherwise (1) could have at most three linearly

* "Projective Differential Geometry of Curved Surfaces," first memoir, *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 246-7.

† Loc. cit., p. 260.

‡ "Projective Differential Geometry of Curved Surfaces," second memoir, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 79. We shall hereafter refer to this paper as Second memoir.

independent solutions. Hence these points may be used as the vertices of a local tetrahedron of reference for the purpose of studying S in the neighborhood of the point y . An expression of the form

$$\tau = a_1y + a_2y_u + a_3y_v + a_4y_{uv},$$

where a_1, a_2, a_3, a_4 are analytic functions of u and v , assumes four values $\tau^{(1)}, \tau^{(2)}, \tau^{(3)}, \tau^{(4)}$ corresponding to the four values of y . Hence τ determines a point whose local coördinates may be defined by writing

$$x_1 = a_1, \quad x_2 = a_2, \quad x_3 = a_3, \quad x_4 = a_4.$$

Consider the case when two such points

$$(5) \quad \begin{aligned} \tau_1 &= a_1y + a_2y_u + a_3y_v + a_4y_{uv}, \\ \tau_2 &= b_1y + b_2y_u + b_3y_v + b_4y_{uv} \end{aligned}$$

are given for every point y of the surface S . If we associate the line l , determined by τ_1 and τ_2 , with each point of S , these lines form a congruence. Wilczynski and Green have considered such congruences in cases where the lines l pass through the point y or lie in the corresponding tangent plane. In this paper we shall consider the more general problems connected with the congruences determined by the lines l when l has an arbitrary position relative to S . General formulas will be obtained for the torsal curves and the guide curves (to be defined later), and for the focal points on l . These general formulas will then be applied to certain special congruences in connection with which various configurations of the lines themselves will be studied.

§ 2. Determination of the Developables of a Congruence.

For each point y of the surface S the points (5) determine a line l whose homogeneous line coördinates are

$$\omega_{ik} = a_ib_k - a_kb_i \quad (i, k = 1, 2, 3, 4).$$

We wish to find the curves on S along which y must move in order that the corresponding line l of the congruence may describe a developable. These curves will be called the torsal curves of the surface S with respect to the congruence. Let u and v increase by amounts du and dv , where du and dv are infinitesimals, in such a way that the point y will change to $y + dy$, a new point on one of the torsal curves. Then the points τ_1 and τ_2 will move to $\tau_1 + d\tau_1$ and $\tau_2 + d\tau_2$ respectively. The line joining $\tau_1 + d\tau_1$ to $\tau_2 + d\tau_2$ is a generator of the developable consecutive to $\tau_1\tau_2$ and must intersect $\tau_1\tau_2$. Therefore $\tau_1, \tau_2, d\tau_1, d\tau_2$ must be coplanar. Now

$$d\tau_1 = (\tau_1)_u du + (\tau_1)_v dv, \quad d\tau_2 = (\tau_2)_u du + (\tau_2)_v dv.$$

Hence, using (5) and (1),

$$(6) \quad \begin{aligned} d\tau_1 &= (A_1 du + A'_1 dv)y + (A_2 du + A'_2 dv)y_u \\ &\quad + (A_3 du + A'_3 dv)y_v + (A_4 du + A'_4 dv)y_{uv}, \\ d\tau_2 &= (B_1 du + B'_1 dv)y + (B_2 du + B'_2 dv)y_u \\ &\quad + (B_3 du + B'_3 dv)y_v + (B_4 du + B'_4 dv)y_{uv}, \end{aligned}$$

where

$$(7) \quad \begin{aligned} A_1 &= (a_1)_u - fa_2 + (2bg - f_v)a_4, & A'_1 &= (a_1)_v - ga_3 + (2a'f - g_u)a_4, \\ A_2 &= (a_2)_u + a_1 + 4a'ba_4, & A'_2 &= (a_2)_v - 2a'a_3 - (g + 2a'_u)a_4, \\ A_3 &= (a_3)_u - 2ba_2 - (f + 2b_v)a_4, & A'_3 &= (a_3)_v + a_1 + 4a'ba_4, \\ A_4 &= (a_4)_u + a_3, & A'_4 &= (a_4)_v + a_2, \\ B_1 &= (b_1)_u - fb_2 + (2bg - f_v)b_4, & B'_1 &= (b_1)_v - gb_3 + (2a'f - g_u)b_4, \\ B_2 &= (b_2)_u + b_1 + 4a'bb_4, & B'_2 &= (b_2)_v - 2a'b_3 - (g + 2a'_u)b_4, \\ B_3 &= (b_3)_u - 2bb_2 - (f + 2b_v)b_4, & B'_3 &= (b_3)_v + b_1 + 4a'bb_4, \\ B_4 &= (b_4)_u + b_3, & B'_4 &= (b_4)_v + b_2. \end{aligned}$$

A necessary and sufficient condition that the points $\tau_1, \tau_2, d\tau_1, d\tau_2$ lie in a plane is that the determinant of the coördinates of the four points be zero. Expanding the determinant we obtain

$$(8) \quad Ldu^2 + 2Mdudv + Ndv^2 = 0,$$

where

$$\begin{aligned} L &= \omega_{12}(A_3B_4) + \omega_{13}(A_4B_2) + \omega_{14}(A_2B_3) \\ &\quad + \omega_{23}(A_1B_4) + \omega_{42}(A_1B_3) + \omega_{34}(A_1B_2), \\ 2M &= \omega_{12}[(A_3B'_4) + (A'_3B_4)] + \omega_{13}[(A_4B'_2) + (A'_4B_2)] \\ &\quad + \omega_{14}[(A_2B'_3) + (A'_2B_3)] + \omega_{23}[(A_1B'_4) + (A'_1B_4)] \\ &\quad + \omega_{42}[(A_1B'_3) + (A'_1B_3)] + \omega_{34}[(A_1B'_2) + (A'_1B_2)], \\ N &= \omega_{12}(A'_3B'_4) + \omega_{13}(A'_4B'_2) + \omega_{14}(A'_2B'_3) \\ &\quad + \omega_{23}(A'_1B'_4) + \omega_{42}(A'_1B'_3) + \omega_{34}(A'_1B'_2). \end{aligned}$$

Here (A_3B_4) , etc., are the determinantal expressions $A_3B_4 - A_4B_3$, etc. We may then conclude

The torsal curves of the surface with respect to the congruence are determined by (8), a quadratic differential equation which determines a net of curves on the surface S .

§ 3. The Focal Points of the Lines l .

Each line l of the congruence belongs to two developables of the congruence, and the two points in which l touches the cuspidal edges of these developables (viz., the focal points of the line l) will now be found. Any point on the line l is given by an expression of the form

$$(9) \quad \varphi = \lambda\tau_1 + \mu\tau_2,$$

where τ_1 and τ_2 are given by (5) and λ and μ are arbitrary functions of u and v . If φ is to be a focal point of l , then the tangent plane to the surface formed by all the points φ , as u and v vary, must contain the line l . Hence $\tau_1, \tau_2, \varphi_u, \varphi_v$ must be coplanar. Noting that

$$\begin{aligned}\varphi_u &= \lambda_u \tau_1 + \mu_u \tau_2 + \lambda(\tau_1)_u + \mu(\tau_2)_u, \\ \varphi_v &= \lambda_v \tau_1 + \mu_v \tau_2 + \lambda(\tau_1)_v + \mu(\tau_2)_v,\end{aligned}$$

it follows that since τ_1 and τ_2 are coplanar with $\varphi_u - \lambda_u \tau_1 - \mu_u \tau_2$ and $\varphi_v - \lambda_v \tau_1 - \mu_v \tau_2$ they are also coplanar with $\lambda(\tau_1)_u + \mu(\tau_2)_u$ and $\lambda(\tau_1)_v + \mu(\tau_2)_v$. So that a necessary and sufficient condition that these points lie in the same plane is that the determinant of the coördinates of the points $\tau_1, \tau_2, \lambda(\tau_1)_u + \mu(\tau_2)_u, \lambda(\tau_1)_v + \mu(\tau_2)_v$ be zero. Expanding this determinant and using (7) we obtain

$$(10) \quad L'\lambda^2 + 2M'\lambda\mu + N'\mu^2 = 0,$$

where

$$\begin{aligned}L' &= \omega_{12}(A_3A'_4) + \omega_{13}(A_4A'_2) + \omega_{14}(A_2A'_3) \\ &\quad + \omega_{23}(A_1A'_4) + \omega_{42}(A_1A'_3) + \omega_{34}(A_1A'_2), \\ 2M' &= \omega_{12}[(A_3B'_4) + (B_3A'_4)] + \omega_{13}[(A_4B'_2) + (B_4A'_2)] \\ &\quad + \omega_{14}[(A_2B'_3) + (B_2A'_3)] + \omega_{23}[(A_1B'_4) + (B_1A'_4)] \\ &\quad + \omega_{42}[(A_1B'_3) + (B_1A'_3)] + \omega_{34}[(A_1B'_2) + (B_1A'_2)], \\ N' &= \omega_{12}(B_3B'_4) + \omega_{13}(B_4B'_2) + \omega_{14}(B_2B'_3) \\ &\quad + \omega_{23}(B_1B'_4) + \omega_{42}(B_1B'_3) + \omega_{34}(B_1B'_2).\end{aligned}$$

Here we have determinantal quantities similar to those in (8). The two values of λ/μ obtained from (10), substituted in (9), give the focal points of the line l , viz.,

$$(11) \quad \varphi_1 = \lambda_1 \tau_1 + \mu_1 \tau_2, \quad \varphi_2 = \lambda_2 \tau_1 + \mu_2 \tau_2.$$

Their product determines a covariant

$$(12) \quad N'\tau_1^2 - 2M'\tau_1\tau_2 + L'\tau_2^2.$$

We may then conclude

The focal points of the lines l of the congruence are given by the factors of the covariant expression (12).

§ 4. The Guide Curves of the Congruence.

With each point y of the surface S is associated a unique plane through it, viz., the plane containing the line l determined by τ_1 and τ_2 . This plane intersects the tangent plane at y in a well-determined line unless l lies in the tangent plane or passes through y . We shall now find the family of curves on S which will have these lines as tangents. These curves will

be called the *guide curves* of the surface with respect to the given congruence since the line l acts as a guide in determining the position of the plane. The equation of this plane is

$$(13) \quad \omega_{34}x_2 + \omega_{42}x_3 + \omega_{23}x_4 = 0.$$

It intersects the tangent plane $x_4 = 0$ in the line

$$\omega_{34}x_2 + \omega_{42}x_3 = 0, \quad x_4 = 0.$$

If $u = u(t)$, $v = v(t)$ is the equation of the desired curve through y , the condition that this line be tangent to the curve gives

$$(14) \quad \omega_{34}du + \omega_{42}dv = 0$$

as the differential equation of the required curves on S .

Miss Sperry* has developed the differential equation for the union curves on the surface S , which are curves such that the osculating plane of a point y on the curve will contain the generator of a congruence when this generator passes through the point y and does not lie in the tangent plane to S at y . We now desire to find the condition under which the guide curves are also union curves.

As before let the equation of a curve on S be given by $u = u(t)$, $v = v(t)$. The osculating plane of this curve at one of its points y is

$$(15) \quad 2u'v'^2x_2 - 2u'^2v'x_3 - (u''v' - u'v'' + 2bu'^3 - 2a'v'^3)x_4 = 0,$$

where accents indicate differentiation as to t . If (13) and (15) are to represent the same plane,

$$(16) \quad \frac{2u'v'^2}{\omega_{34}} = \frac{-2u'^2v'}{\omega_{42}} = \frac{u'v'' - v'u'' - 2bu'^3 + 2a'v'^3}{\omega_{23}}.$$

$u' = 0$, $v' = 0$ is a solution of these equations but in this case the coordinates of the point y' would be $(0, 0, 0, 0)$ which is not admissible. Assuming $u' \neq 0$ and introducing u as the parameter instead of t we obtain

$$(17) \quad \frac{dv}{du} = \frac{\omega_{34}}{\omega_{42}}, \quad 2\omega_{23}\left(\frac{dv}{du}\right)^2 = \omega_{34}\left[\frac{d^2v}{du^2} - 2b + 2a'\left(\frac{dv}{du}\right)^3\right].$$

Substituting the value of dv/du from the first equation of (17), which is the same as (14), into the second we find

$$(18) \quad 2\omega_{23}\omega_{34}\omega_{42} - \omega_{34}\omega_{42}(\omega_{42})_u + \omega_{42}^2(\omega_{34})_u + \omega_{34}^2(\omega_{42})_v - \omega_{42}\omega_{34}(\omega_{34})_v + 2b\omega_{42}^3 + 2a'\omega_{44}^3 = 0$$

* "Properties of a Certain Projectively Defined Two-parameter Family of Curves on a General Surface," *American Journal of Mathematics*, Vol. XL (1918), pp. 213-224.

as the relation that must be satisfied by the line coördinates of l in order that the guide curves may be union curves.

Condition (18) is satisfied by $\omega_{34} = \omega_{42} = 0$. Two cases may arise depending on whether $\omega_{23} = 0$ or $\omega_{23} \neq 0$. If $\omega_{23} = 0$, we see from (16) that the solution is $u' = 0$, $v' = 0$, which has been excluded. Geometrically this occurs when the line l passes through the point y . As stated above this case has been considered by Miss Sperry from another point of view. If $\omega_{23} \neq 0$, the solution of (16) is $u' = 0$ if $v' \neq 0$ and is $v' = 0$ if $u' \neq 0$. This is the case when the line l lies in the tangent plane to S at y and does not pass through y .

We may then conclude

The one-parameter family of guide curves of the surface with respect to the given congruence has (14) for its differential equation. In case (18) is satisfied the guide curves are also union curves. When l passes through the point y on the surface S or lies in the tangent plane to S at y , the guide curves are indeterminate.

§ 5. The Osc-scroll-flec Congruences.

We shall now apply these results to some special congruences closely associated with a given surface. We begin by recalling what Wilczynski calls the osculating ruled surfaces of the first and second kinds, respectively. One of these, R_1 , is the locus of the tangents to the asymptotic curves $v = \text{const.}$ along a fixed curve $u = \text{const.}$, and R_2 is the locus of the tangents to the asymptotic curves $u = \text{const.}$ along a fixed curve $v = \text{const.}$ The differential equations of R_1 and R_2 referred to our local tetrahedron of reference are given in Wilczynski's second memoir, pp. 81-82.

We shall have occasion to use the invariants of weights four, nine, and ten for R_1 and R_2 . Expressed in terms of the coefficients of (1), for R_1 they are*

$$\begin{aligned} \theta_4 &= 2^6(a_u'^2 - 2a' a_{uu}' - 4a'^2 f - 4a'^2 b_v), \\ \theta_9 &= 4(C^3 + 8a' C C_{uv} - 8a' C_u C_v), \\ \theta_{10} &= 2^6 a'^2 C_u^2 - C^2 \theta_4, \end{aligned} \quad (19)$$

where†

$$C = 8a'_{uv} - 8 \frac{a'_u a'_v}{a'} - 32a'^2 b. \quad (20)$$

In obtaining θ_{10} and θ_9 use has been made of the relation

$$2a'_v \theta_4 - a'(\theta_4)_v = 16a'^2 C_u. \quad (21)$$

The corresponding invariants θ'_4 , θ'_9 , θ'_{10} for R_2 are obtained from (19) by

* Second memoir, pp. 81, 84. θ_9 and θ_{10} were obtained by C. D. Meacham, a student at the University of Chicago.

† Loc. cit., p. 84.

the transpositions

$$(22) \quad (a'b), \quad (fg), \quad (uv), \quad (\theta_4\theta'_4), \quad (CC'),$$

where C' is obtained from C by the same transpositions.

The vanishing of θ_4 is the condition that R_1 have coincident branches to its flecnode curve, and $\theta'_4 = 0$ is the corresponding condition for R_2 . Consider for the present the case where $\theta_4 \neq 0$. Then R_1 may be referred to its flecnode curve. Let us denote the flecnodes on the generator yy_u of R_1 by η and ζ and let the lines ηr and ζs^* be the flecnode tangents of R_1 at η and ζ respectively. To every point y of S there belongs such a line ηr and hence we obtain a congruence associated with S . Similarly ζs generates a second congruence. Relative to the osculating ruled surface R_2 two other congruences are determined in this way provided that the branches of the flecnode curve of R_2 do not coincide, i.e., provided $\theta'_4 \neq 0$. We shall call these four congruences the *osc-scroll-flec congruences*, and we shall designate the congruences determined by ηr and ζs by Γ_1 and Γ'_1 respectively and the corresponding congruences associated with R_2 by Γ_2 and Γ'_2 . Referred to our local tetrahedron of reference, η, r, ζ, s are given by†

$$(23) \quad \begin{aligned} -32a'\sqrt{\theta_4}\eta &= (8a'_u - \sqrt{\theta_4})y - 16a'y_u, \\ -32a'\sqrt{\theta_4}r &= 64a'^2by + 2(8a'_u - \sqrt{\theta_4})y_v - 32a'y_{uv}, \\ 32a'\sqrt{\theta_4}\zeta &= (8a'_u + \sqrt{\theta_4})y - 16a'y_u, \\ 32a'\sqrt{\theta_4}s &= 64a'^2by + 2(8a'_u + \sqrt{\theta_4})y_v - 32a'y_{uv}. \end{aligned}$$

Hence, for the congruence Γ_1 , the two points τ_1 and τ_2 of our general theory are given by

$$(24) \quad \begin{aligned} \tau_1 &= (8a'_u - \sqrt{\theta_4})y - 16a'y_u, \\ \tau_2 &= 64a'^2by + 2(8a'_u - \sqrt{\theta_4})y_v - 32a'y_{uv}. \end{aligned}$$

The corresponding formulas for the congruence Γ'_1 differ from (24) merely in the sign of the radical.

Substituting (24) in (8) of § 2 and making use of (2), (19), (20), (21) we find that the torsal curves for Γ_1 are given by

$$(25) \quad \begin{aligned} [(\theta_4)_u^2 - 16a'^2\theta_4\theta'_4]du^2 - 4(\theta_4)_u(8a'C_u + C\sqrt{\theta_4})dudv \\ + 4(8a'C_u + C\sqrt{\theta_4})^2dv^2 = 0. \end{aligned}$$

By changing the sign of the radical in (25) we obtain the torsal curves for

* Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces." Teubner, Leipzig, 1906, p. 124. We shall hereafter refer to this work as "Proj. Diff. Geom." The quantities r and s are there referred to as ρ and σ .

† Second memoir, p. 84.

Γ'_1 , viz.,

$$(26) \quad [(\theta_4)_u^2 - 16a'^2\theta_4\theta'_4]du^2 - 4(\theta_4)_u(8a'C_u - C\sqrt{\theta_4})dudv + 4(8a'C_u - C\sqrt{\theta_4})^2dv^2 = 0.$$

The torsal curves for Γ_2 and Γ'_2 may be obtained from (25) and (26) by means of the transpositions (22).

When $\theta_4 = 0$, R_1 has but one branch to its flecnode curve. In that case the single flecnode tangent may be determined by*

$$\begin{aligned} 2a'\zeta &= a'_uy - 2a'y_u, \\ 2a's &= 8a'^2by + 2a'_uy_v - 4a'y_{uv}, \end{aligned}$$

where ζ is the flecnode. Γ_1 and Γ'_1 coincide and the torsal curves for this congruence are found to be

$$(27) \quad 4a'^2\theta'_4du^2 - C^2dv^2 = 0,$$

a conjugate net on S .

The focal points of the osc-scroll-flec congruences are found by factoring the covariant expression (12) of § 3. For Γ_1 when $\theta_4 \neq 0$, this covariant is

$$(28) \quad (64b_v^2 - \theta'_4)\tau_1^2 - 128bb_v\tau_1\tau_2 + 64b^2\tau_2^2,$$

aside from a factor $8a'C_u + C\sqrt{\theta_4}$ which is, in general, different from zero. Its vanishing makes $\theta_{10} = 0$. But $\theta_4 \neq 0$, $\theta_{10} = 0$ is the condition that R_1 have a straight line directrix.† An examination of the equations of the ruled surface R_1 when referred to its flecnode curves‡ shows that the flecnode curve C_η is also an asymptotic curve on R_1 and consequently is a straight line,§ the straight line directrix of R_1 . Equation (25) shows that in this case the torsal net reduces to $du^2 = 0$. As y moves along $u = \text{const.}$, R_1 remains the same, the flecnode curve is a straight line and is, in fact, the line ηr ; hence we see that the developable generated by ηr reduces to a straight line and consequently the congruence Γ_1 degenerates into a ruled surface.

For the congruence Γ'_1 we obtain the same expression (28) for the focal points, but with the factor $8a'C_u - C\sqrt{\theta_4}$ omitted instead of $8a'C_u + C\sqrt{\theta_4}$. Its vanishing would cause θ_{10} to vanish and Γ'_1 would degenerate into a ruled surface. If both of the factors of θ_{10} vanish, R_1 will have two straight line directrices and hence will belong to a linear congruence, and Γ_1 and Γ'_1 will degenerate into ruled surfaces.

* Second memoir, p. 86.

† "Proj. Diff. Geom.," p. 167.

‡ Second memoir, pp. 83-84.

§ "Proj. Diff. Geom.," p. 150.

When $\theta_4 = 0$, the congruences Γ_1 , Γ'_1 coincide. The focal points are given by (28) aside from a factor C which is, in general, non-vanishing. If $C = 0$, the congruence degenerates into a ruled surface.

The focal points for the congruences Γ_2 and Γ'_2 , whether distinct or coincident, are obtained from (28) by means of the transpositions (22).

From the theorem of § 4 we find that the guide curves are given by $v = \text{const.}$ for Γ_1 and Γ'_1 , and by $u = \text{const.}$ for Γ_2 and Γ'_2 , as is obvious geometrically. The quantity $\omega_{34} = 0$ while $\omega_{42} \neq 0$, for Γ_1 and Γ'_1 , and $\omega_{42} = 0$, $\omega_{34} \neq 0$ for Γ_2 and Γ'_2 , hence the condition that the guide curves be union curves is not satisfied and the union curves do not exist.

In order to interpret geometrically special cases arising out of the discussion of the equations which have just been found, it will be convenient to give some geometric properties which follow when certain of the invariants are zero.

As previously mentioned, $\theta_4 = 0$ is the condition that R_1 may have coincident branches to its flecnode curve. It is also the condition that the focal sheets of the osc-scroll-flec congruences associated with R_2 coincide, as may be seen from the equation obtained from (28) by (22). In this case the cuspidal edges on this focal sheet are asymptotic curves on that surface. We also noted that $\theta_4 \neq 0$, $\theta_{10} = 0$ is the condition for R_1 to have a straight line directrix. Corresponding conditions hold when $\theta'_4 = 0$ and when $\theta'_4 \neq 0$, $\theta'_{10} = 0$. The conditions $C = 0$ and $C' = 0$ correspond to the cases when the asymptotic curves $v = \text{const.}$ and $u = \text{const.}$, respectively, belong to linear complexes.* If $C = C' = 0$, $\theta_4 \neq 0$, $\theta'_4 \neq 0$, R_1 and R_2 belong to linear congruences with distinct directrices. If $C = \theta_4 = 0$, $C' = \theta'_4 = 0$, these linear congruences have coincident directrices.†

By an examination of the discriminant of (25) for $\theta_4 \neq 0$, and of (27) for $\theta_4 = 0$, we find that the torsal curves for Γ_1 represent a one-parameter family of curves instead of a proper net in a number of special cases, which may be interpreted geometrically in the light of the foregoing properties. Among these special cases we find several where the asymptotic curves on S are torsal curves. When $u = \text{const.}$ is a torsal curve, Γ_1 degenerates into a ruled surface. When $v = \text{const.}$ is a torsal curve, the focal sheets of Γ_1 coincide. An examination of (25) shows that the asymptotic curves of S cannot both be torsal curves under the same conditions. Furthermore, the torsal curves form conjugate nets under special conditions which may be interpreted geometrically. Corresponding conditions hold for the torsal curves of Γ'_1 and we can readily find the conditions under which these

* C. T. Sullivan, "Properties of Surfaces whose Asymptotic Curves belong to Linear Complexes," *Transactions of the American Mathematical Society*, Vol. 15 (1914), p. 178.

† Second memoir, p. 86.

curves coincide with the torsal curves of Γ'_1 . Similar conditions hold relative to the torsal curves of the Γ_2 and Γ'_2 congruences.

We shall now return to some general considerations regarding the osc-scroll-flec congruences. The tangents to the two torsal curves of Γ_1 at a point y of the surface are the lines joining y to the two points

$$y_u + (dv/du)_1 y_v, \quad y_u + (dv/du)_2 y_v,$$

where $(dv/du)_1$ and $(dv/du)_2$ are the two roots of equation (25) regarded as a quadratic in dv/du . Let t_1 and t_2 be these two tangents. The directions conjugate to t_1 and t_2 are obtained by joining the point y to the points

$$y_u - (dv/du)_1 y_v, \quad y_u - (dv/du)_2 y_v,$$

respectively. Using the term employed by Green* we shall call these two new tangents the reflected tangents of t_1 and t_2 . The totality of these reflected tangents determines a new net on S which may, with Green, be called the reflected Γ_1 -curves. They are determined by the differential equation

$$(29) \quad [(\theta_4)_u^2 - 16a'^2\theta_4\theta'_4]du^2 + 4(\theta_4)_u(8a'C_u + C\sqrt{\theta_4})dudv \\ + 4(8a'C_u + C\sqrt{\theta_4})^2dv^2 = 0,$$

an equation differing from (25) in the sign of the middle term only, since its roots are those of (25) with the signs changed. The torsal curves of Γ_1 , the reflected Γ_1 -curves, and the asymptotic curves at any point y of the surface S thus constitute three pairs in an involution. The Jacobian of the torsal curves of Γ_1 and the reflected Γ_1 -curves gives the double elements of the involution, which constitute, of course, a pair of conjugate tangents, namely

$$(30) \quad [(\theta_4)_u^2 - 16a'^2\theta_4\theta'_4]du^2 - 4(8a'C_u + C\sqrt{\theta_4})^2dv^2 = 0.$$

We may then say

The torsal curves of each osc-scroll-flec congruence, the corresponding reflected curves, and the asymptotic curves at any point y of the surface S constitute three pairs in an involution. The double elements of the involutions so determined give four unique projectively defined conjugate nets on S , one relative to each of the osc-scroll-flec congruences. Their differential equations are given by (30) and the equations obtained from (30) by changing the sign of the radical and by the transpositions (22) applied to these two equations.

The focal points of the line ηr for the Γ_1 -congruence were found to be given by the expressions

$$\varphi_1 = \lambda_1\tau_1 + \mu_1\tau_2, \quad \varphi_2 = \lambda_2\tau_1 + \mu_2\tau_2,$$

* Memoir on the general theory of surfaces and rectilinear congruences. *Transactions of the American Mathematical Society*, Vol. 20 (1919), p. 93.

where λ_1/μ_1 and λ_2/μ_2 are the roots of the quadratic equation

$$64b^2\lambda^2 + 128bb_v\lambda\mu + (64b_v^2 - \theta_4')\mu^2 = 0,$$

provided $\theta_{10} \neq 0$. A point which proves to be of considerable interest is obtained by finding the harmonic conjugate of η , i.e., τ_1 , with respect to φ_1 and φ_2 . It is given by

$$\alpha = 1/2(\lambda_1/\mu_1 + \lambda_2/\mu_2)\tau_1 + \tau_2.$$

From the above quadratic equation in λ/μ and from (24) this is found to be the point

$$(31) \quad \alpha = (-8a_u'b_v + 64a'^2b^2 + b_v\sqrt{\theta_4})y + 16a'b_vy_u \\ + 2b(8a_u' - \sqrt{\theta_4})y_v - 32a'by_{uv}.$$

The corresponding point on ζs of the congruence Γ'_1 is given by

$$(32) \quad \beta = (-8a_u'b_v + 64a'^2b^2 - b_v\sqrt{\theta_4})y + 16a'b_vy_u \\ + 2b(8a_u' + \sqrt{\theta_4})y_v - 32a'by_{uv}.$$

Relative to Γ_2 and Γ'_2 we obtain the two points α' and β' , found from (31) and (32) by the transpositions (22).

The equation of the osculating quadric Q of the surface S at the point y is*

$$(33) \quad x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0.$$

It can readily be shown that the lines $\alpha\beta$ and $\alpha'\beta'$ lie on this quadric and that they intersect one another at the point where the directrix of the second kind d' † intersects the osculating quadric. $\alpha\beta$ intersects the side yy_u of the local tetrahedron of reference and $\alpha'\beta'$ intersects the side yy_v .

For each point y on S a line $\alpha\beta$ is determined even when $\theta_4 = 0$. If $\theta_4 = 0$, α and β coincide but the line $\alpha\beta$ is then the line joining $\alpha = \beta$ to the point where d' intersects the osculating quadric. Similarly for $\alpha'\beta'$. Consequently we have two new congruences associated with S . The torsal curves for these congruences are given by

$$(34) \quad 2^8b^2C'^2du^2 - 2^5b\theta_4'(C' + 32a'b^2)dudv + (\theta_4'^2 - 2^{10}b^4\theta_4)dv^2 = 0, \\ (\theta_4^2 - 2^{10}a'^4\theta_4')du^2 - 2^5a'\theta_4(C + 32a'^2b)dudv + 2^8a'^2C^2dv^2 = 0,$$

respectively. If $\theta_4' = 0$, the torsal curves given in the first equation of (34) determine a conjugate net which coincides with the torsal net for the coincident Γ_2 and Γ'_2 congruences. In this case the focal sheets of Γ_1 and Γ'_1 coincide and $\alpha\beta$ becomes the line joining the coincident focal points on ηr and ζs . The line $\alpha'\beta'$ is now the line joining $\alpha' = \beta'$ to the point where d'

* Second memoir, p. 82.

† Loc. cit., p. 97.

intersects the osculating quadric. Similar conditions hold if $\theta_4 = 0$. If these conditions occur simultaneously, i.e., if $\theta'_4 = \theta_4 = 0$, the torsal curves of the two congruences reduce to $du^2 = 0$ and to $dv^2 = 0$, respectively.

The focal points on $\alpha\beta$ and $\alpha'\beta'$ are given by the factors of the covariant expressions

$$(35) \quad \begin{aligned} &2^{10}b^2C'\beta^2 - 2^8a'b^2\theta'_4\alpha\beta - (a'\theta_4'^2 + 16b^2C'\theta_4)\alpha^2, \\ &2^{10}a'^2C\beta'^2 - 2^8a'^2b\theta_4\alpha'\beta' - (b\theta_4^2 + 16a'^2C\theta'_4)\alpha'^2, \end{aligned}$$

respectively. For $\theta'_4 = 0$, the focal points on $\alpha\beta$ separate α and β harmonically. Similarly in the case $\theta_4 = 0$, α' and β' are separated harmonically by the focal points of the line $\alpha'\beta'$. When $\theta_4 = \theta'_4 = 0$, the focal points of $\alpha\beta$ are coincident with β and the focal points of $\alpha'\beta'$ are coincident with β' , i.e., the focal sheets of both of these congruences coincide.

The guide curves for $\alpha\beta$ and $\alpha'\beta'$ are found to be $u = \text{const.}$ and $v = \text{const.}$, respectively, as is obvious geometrically.

§ 6. The Congruences Determined by the Pairs of Complexes C_1 , C' and C_2 , C'' .

Associated with a point y of the surface S there are four complexes,—the two complexes C_1 and C_2 which osculate the ruled surfaces R_1 and R_2 respectively and the two complexes C' and C'' which osculate the asymptotic curves of the first and second kinds respectively. Four of the pairs of complexes obtained from these are in involution, i.e., their bilinear invariants are zero.* Professor Wilczynski has considered† more in detail the congruences obtained from the directrices of the congruence common to the osculating complexes C' and C'' . These he called the directrix congruences of the first and second kinds. We shall proceed to consider the other three pairs which are in involution. They are the pairs C_1 , C' ; C_2 , C'' ; and C_1 , C_2 . The first two pairs may be considered together since they are symmetrically situated with respect to the surface S .

The equations of the complexes C_1 and C_2 referred to our local tetrahedron of reference are given by‡

$$(36) \quad C_1 : a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{34}\omega_{34} + a_{42}\omega_{42} = 0,$$

where

$$\begin{aligned} a_{12} &= 0, & a_{13} &= 2^8a'^2C, & a_{14} &= a_{23} = 2^7a'(a'C_u + a'_u C), \\ a_{34} &= -2^9a'^3bC, & a_{42} &= -[C(\theta_4 + 64a_u'^2) + 2^7a'a'_u C_u], \end{aligned}$$

* Second memoir, p. 95.

† Loc. cit., pp. 114–120.

‡ Loc. cit., pp. 85, 86, 89.

with the invariant

$$(37) \quad A = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 2^8 a'^2 \theta_{10};$$

and

$$(38) \quad C_2 : b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{34}\omega_{34} + b_{42}\omega_{42} = 0,$$

where

$$\begin{aligned} b_{12} &= 2^8 b^2 C', & b_{13} &= 0, & b_{14} &= -b_{23} = 2^7 b(bC'_v + b_v C'), \\ b_{34} &= C'(\theta'_4 + 64b_v^2) + 2^7 b b_v C'_v, & b_{42} &= 2^9 a' b^3 C', \end{aligned}$$

with the invariant

$$(39) \quad B = b_{12}b_{34} + b_{13}b_{42} + b_{14}b_{23} = -2^8 b^2 \theta'_{10}.$$

In writing down the above coefficients and invariants use has been made of (19) and (21). The equations of the complexes C' and C'' are*

$$(40) \quad C' : -b_v \omega_{34} - b \omega_{14} + b \omega_{23} = 0,$$

with the invariant $A' = -b^2$,

$$(41) \quad C'' : -a'_u \omega_{42} + a' \omega_{14} + a' \omega_{23} = 0,$$

with the invariant $A'' = a'^2$. It is to be noted that (38), (39) and (41) follow from (36), (37) and (40) respectively by applying the transpositions (22) and by interchanging the subscripts 2 and 3.

Consider the congruence determined by the linear complexes C_1 and C' . It will have two directrices, d_1 and d'_1 , say. For every surface point we have two lines so determined. Hence d_1 and d'_1 will determine congruences associated with the surface S . We shall speak of these as the d_1 -congruence and the d'_1 -congruence. The equations of these directrices referred to the local tetrahedron of reference are found to be†

$$(42) \quad \begin{aligned} &a_{13}x_3 + (a_{14} + 16a'\sqrt{\theta_{10}})x_4 = 0, \\ &ba_{13}x_1 + b(a_{14} - 16a'\sqrt{\theta_{10}})x_2 - (ba_{34} + 16a'b_v\sqrt{\theta_{10}})x_4 = 0, \end{aligned}$$

for d_1 and

$$(43) \quad \begin{aligned} &a_{13}x_3 + (a_{14} - 16a'\sqrt{\theta_{10}})x_4 = 0, \\ &ba_{13}x_1 + b(a_{14} + 16a'\sqrt{\theta_{10}})x_2 - (ba_{34} - 16a'b_v\sqrt{\theta_{10}})x_4 = 0, \end{aligned}$$

for d'_1 . In finding (42) and (43) it is useful to note that

$$a_{13}a_{24} - a_{14}^2 = 2^8 a'^2 \theta_{10}.$$

* Second memoir, pp. 92, 94.

† In obtaining these equations use has been made of equations (67)–(69) on pp. 94–95 of Second memoir where (69) should read $A''\omega^2 - (A', A'')\omega + A' = 0$.

Since equations (43) differ from (42) only in the sign of the radical, we shall discuss the d_1 -congruence in detail and obtain results for the other by changing the sign of $\sqrt{\theta_{10}}$.

As the two points (5) determining d_1 we may take the points of intersection of d_1 with the planes $x_2 = 0$ and $x_3 = 0$. d_1 intersects the edge yy_u of the local tetrahedron of reference, as must obviously be the case since yy_u is a line of both C_1 and C' . After substituting from (36) we have

$$(44) \quad \begin{aligned} \tau_1 &= (8a'C_u + 8a'_u C - \sqrt{\theta_{10}})y - 16a'Cy_u, \\ \tau_2 &= (32a'^2b^2C - b_v\sqrt{\theta_{10}})y \\ &\quad + (8a'bC_u + 8ba'_u C + b\sqrt{\theta_{10}})y_v - 16a'bCy_{uv}. \end{aligned}$$

Let τ_1 and σ_1 be the points in which d_1 intersects the osculating quadric Q whose equation is given in (33). τ_1 is given in (44) and

$$(45) \quad \begin{aligned} \sigma_1 &= [8(a'b_v C_u + a'_u b_v C - 8a'^2b^2C) + b_v\sqrt{\theta_{10}}]y - 16a'b_v Cy_u \\ &\quad - 2b(8a'C_u + 8a'_u C + \sqrt{\theta_{10}})y_v + 32a'bCy_{uv}. \end{aligned}$$

Let τ'_1 and σ'_1 be the points in which d'_1 intersects Q . The coördinates of τ'_1 and σ'_1 are obtained from those of τ_1 and σ_1 respectively by changing the sign of $\sqrt{\theta_{10}}$. It can readily be shown that the lines $\sigma_1\sigma'_1$, $\tau_1\sigma'_1$, and $\tau'_1\sigma_1$ lie on the quadric. Moreover, the line $\sigma_1\sigma'_1$ coincides with the line $\alpha\beta$ of § 5. It can easily be verified that τ_1 and τ'_1 are the complex points* on yy_u and hence are harmonically separated by η and ζ , the flecnodes on yy_u . Then since $\eta\tau$, $\zeta\sigma$, $\tau_1\sigma'_1$ and $\tau'_1\sigma_1$ are all generators of the same kind on the osculating quadric, σ_1 and σ'_1 are harmonically separated by α and β . In § 5 we saw that $\alpha\beta$, i.e., $\sigma_1\sigma'_1$, passes through the point where the directrix d' of the second kind intersects the osculating quadric. Similar conditions hold relative to the lines determined in the same way from the complexes C_2 , C'' .

The torsal curves for the d_1 -congruence are found, except for a non-vanishing factor $C/a'\theta_4\theta_{10}$, to be

$$(46) \quad L_1 du^2 + 2M_1 du dv + N_1 dv^2 = 0,$$

where

$$\begin{aligned} L_1 &= b^2C[C(\theta_4P + 32a'^2(\theta_4)_u C_u)^2 + 16a'^2C\theta_{10}(16a'^2\theta_4\theta'_4 - (\theta_4)_u^2) \\ &\quad - 2^8a'^3C'\theta_4\theta_{10}\sqrt{\theta_{10}}], \\ 2M_1 &= 4a'b\theta_4[4a'C'\theta_{10}^2 - bC\theta_9(\theta_4P + 32a'^2(\theta_4)_u C_u)], \\ N_1 &= a'^2\theta_4[4b^2\theta_4\theta_9^2 - \theta_{10}^2\theta'_4 - 16b^2C\theta_4\theta_{10}\sqrt{\theta_{10}}], \end{aligned}$$

in which

$$P = C\theta_4 - 2^6a'^2C_{uu} - 2^6a'a'_u C_u.$$

* "Proj. Diff. Geom.," p. 208.

The torsal curves for the d'_1 -congruence are obtained from (46) by changing the sign of θ_{10} .

Since M_1 does not contain $\sqrt{\theta_{10}}$, we have by subtracting this equation from (46) and dividing by a non-vanishing factor

$$(47) \quad 16a'C'du^2 + \theta_4dv^2 = 0,$$

a conjugate net on S . *It is the only conjugate net of the involution determined by the torsal curves of the d_1 - and d_2 -congruences.* There is a corresponding conjugate net relative to C_2 and C'' . At any point on S the four tangents determined by these two nets are harmonically related if $2^8a'bCC' + \theta_4\theta'_4 = 0$.

The focal points on d_1 are given by the factors of the covariant expression

$$(48) \quad L_1'\tau_2^2 - 2M_1'\tau_1\tau_2 + N_1'\tau_1^2,$$

where

$$L_1' = 2^8CD, \quad 2M_1' = 2^8C(b_vD + E),$$

$$N_1' = 2^7b_vCE + C\theta'_4(D + 8a'T\sqrt{\theta_{10}}) + 2^6b_v^2CD - 2^5a'bC'\theta_9(\theta_{10} + 8a'C_u\sqrt{\theta_{10}}),$$

in which

$$T = 8a'b\theta_9 + 2C_uP + \theta_uC^2, \quad D = 2^8a'^3bC_u\theta_9 + \theta_{10}P - 4a'T\sqrt{\theta_{10}},$$

$$E = a'bC\theta_9(\theta_4)_u - 2a'^2C\theta'_4\theta_{10} + a'C'\theta_{10}\sqrt{\theta_{10}}.$$

The focal points on d'_1 are obtained from (48) by changing the sign of $\sqrt{\theta_{10}}$.

The guide curves for the d_1 - and d'_1 -congruences are given by $v = \text{const.}$ and these curves are not union curves.

The torsal curves, the focal points, and the guide curves relative to the complexes C_2 and C'' may be obtained at once by means of the transpositions (22) together with the transposition $(\theta_{10}, \theta'_{10}), (\theta_9, \theta'_9)$.

§ 7. The Congruences Determined by the Pair of Complexes C_1 and C_2 .

The equations of the linear complexes C_1 and C_2 , whose bilinear invariant (A, B) is zero, are given in (36) and (38). The following relations

$$(49) \quad a_{13}b_{24} = b_{12}a_{34}, \quad a'^2\theta_{10}(b_{12}b_{34} - b_{14}^2) = b^2\theta'_{10}(a_{13}a_{24} - a_{14}^2),$$

between θ_{10} and θ'_{10} and the coefficients of (36) and (38) are useful in obtaining further results. As in the case of § 6 the equations of the two directrices of the congruence determined by C_1 and C_2 are given by

$$(50) \quad -\epsilon a'b_{12}\sqrt{\theta_{10}}x_2 + ba_{13}\sqrt{\theta'_{10}}x_3 + (ba_{14}\sqrt{\theta'_{10}} - \epsilon a'b_{14}\sqrt{\theta_{10}})x_4 = 0,$$

$$\epsilon a'b_{12}\sqrt{\theta_{10}}x_1 + (ba_{14}\sqrt{\theta'_{10}} + \epsilon a'b_{14}\sqrt{\theta_{10}})x_3 + (ba_{24}\sqrt{\theta'_{10}} - \epsilon a'b_{24}\sqrt{\theta_{10}})x_4 = 0,$$

where $\epsilon = \pm 1$. Let δ_1 be the directrix determined by using $\epsilon = +1$ in

(50) and δ_2 the directrix corresponding to $\epsilon = -1$. It can be shown readily that *the points of intersection of δ_1 and δ_2 with the coördinate planes $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ are harmonic conjugates with respect to the osculating quadric Q .*

For the points τ_1 and τ_2 determining δ_1 we may take the points of intersection of δ_1 with the planes $x_2 = 0$ and $x_3 = 0$. The expressions for these points are

$$\begin{aligned} \tau_1 &= (ba_{24}\sqrt{\theta'_{10}} - a'b_{24}\sqrt{\theta_{10}})y \\ &\quad - (ba_{14}\sqrt{\theta'_{10}} - a'b_{14}\sqrt{\theta_{10}})y_u - a'b_{12}\sqrt{\theta_{10}}y_{uv}, \\ (51) \quad \tau_2 &= - (ba_{34}\sqrt{\theta'_{10}} - a'b_{34}\sqrt{\theta_{10}})y \\ &\quad + (ba_{14}\sqrt{\theta'_{10}} - a'b_{14}\sqrt{\theta_{10}})y_v - ba_{13}\sqrt{\theta'_{10}}y_{uv}. \end{aligned}$$

An examination of the elements entering into the equation of the torsal curves of the congruence determined by δ_1 shows that we obtain the same equation over again if we apply the transformation (22) to it. In this sense we may speak of the torsal curves of the δ_1 -congruence as symmetric. Similar conditions hold for the δ_2 -congruence. These nets may be represented by

$$\begin{aligned} &(\alpha_1\sqrt{\theta_{10}} + \alpha_2\sqrt{\theta'_{10}})du^2 + 2(\beta_1\sqrt{\theta_{10}} + \beta_2\sqrt{\theta'_{10}})dudv \\ &\quad + (\gamma_1\sqrt{\theta_{10}} + \gamma_2\sqrt{\theta'_{10}})dv^2 = 0, \\ (52) \quad &(-\alpha_1\sqrt{\theta_{10}} + \alpha_2\sqrt{\theta'_{10}})du^2 + 2(-\beta_1\sqrt{\theta_{10}} + \beta_2\sqrt{\theta'_{10}})dudv \\ &\quad + (-\gamma_1\sqrt{\theta_{10}} + \gamma_2\sqrt{\theta'_{10}})dv^2 = 0, \end{aligned}$$

where $\gamma_1, \gamma_2, \beta_2$ are obtained from $\alpha_2, \alpha_1, \beta_1$ respectively by (22).

If $\theta_{10} = 0$, $\theta'_{10} \neq 0$, the invariant of the complex C_1 is zero and hence C_1 is a special linear complex unless $\theta_{10} = 0$ by virtue of C being zero in which case the complex C_1 is indeterminate. Excluding that case we see that the congruence determined by C_1 and C_2 has two coincident straight line directrices.* This directrix is given by the two points whose coördinates are

$$(53) \quad (a_{24}, -a_{14}, 0, 0), \quad (a_{34}, 0, -a_{14}, a_{13}).$$

It is easy to verify that this line lies on the osculating quadric since $\theta_{10} = 0$. In the case we are considering we have but one congruence associated with the surface S instead of two. Its torsal curves may be obtained from (8) without a great deal of algebraic work. It is obvious geometrically that its guide curves are $v = \text{const.}$ since the directrix intersects the line yy_u .

If $\theta'_{10} = 0$, $C' \neq 0$, $\theta_{10} \neq 0$, we obtain a congruence generated by the line determined by the points whose coördinates are

$$(54) \quad (b_{34}, 0, -b_{14}, 0), \quad (b_{24}, -b_{14}, 0, b_{12}).$$

* Second memoir, p. 163.

This line lies on the osculating quadric since $\theta'_{10} = 0$ and it intersects the line yy_v . Consequently the guide curves of the congruence determined by it are $u = \text{const.}$

If $\theta_{10} = 0$, $\theta'_{10} = 0$, $C \neq 0$, $C' \neq 0$, the two complexes C_1 and C_2 are both special and the congruence determined by them degenerates into two systems of ∞^2 lines, viz., all the lines in the plane of the two axes of the special complexes and all the lines through their point of intersection.* This point of intersection is the point in which the lines (53) and (54) meet since in this case (53) and (54) are the two axes which are generators of different kinds on the osculating quadric.

Returning to the general case we find that the guide curves of the δ_1 -congruence are given by

$$(55) \quad ba_{13}\sqrt{\theta'_{10}}du - a'b_{12}\sqrt{\theta_{10}}dv = 0,$$

and of the δ_2 -congruence by

$$(56) \quad ba_{13}\sqrt{\theta'_{10}}du + a'b_{12}\sqrt{\theta_{10}}dv = 0.$$

Hence the guide curves of the surface S with respect to these two congruences together constitute a projectively defined conjugate net on S , namely, the net

$$(57) \quad a'^2 C^2 \theta'_{10} du^2 - b^2 C'^2 \theta_{10} dv^2 = 0.$$

* Loc. cit., p. 163.

VITA

Claribel Kendall was born in Denver, Colorado. Her early education was obtained in the public schools of Denver. She was graduated from the University of Colorado with the degrees B.A. and B.E. in 1912, and she received the master's degree in Mathematics from the same institution in 1914. After obtaining this degree she did graduate work with Dr. B. F. Finkel and Dr. A. Cohen during summer sessions at the University of Colorado. The summers of 1915 and 1918 were spent at the University of Chicago, taking graduate work with Professors Moore, Wilczynski, Dickson, and Bliss. She held a scholarship both summers. The summer of 1920 she returned to Chicago, having been granted a fellowship for the year 1920-1921. After five quarters of consecutive work she received the degree Ph.D. in Mathematics at the September convocation of 1921.

She would like to take this opportunity of expressing her appreciation and indebtedness to Professor Wilczynski for his guidance in the preparation of her doctor's dissertation. His interest and encouragement were a constant source of inspiration.

